Universal Probability

See page 467 of the book. There, universal probability is wrongly defined.

Of course, given n independent tosses of a fair coin, the probability of any binary data vector $\mathbf{x} = (x_1, \ldots, x_n)$ thus produced is $1/2^n$. Typically, these vectors are not compressible, their complexity $K(\mathbf{x})$ being n. Quite a few of them are hardly compressible, meaning that their Kolmogorov complexity is a bit less than n. However, there is also (heavily) compressible data, and in general the correct definition of a data vector's $\mathbf{x}$ complexity is as follows (Cover und Thomas 2006: 466, 481, 490):

Let $\mathcal{U}$ be a universal computer (e.g. a Turing machine) and let $\mathcal{U}(\mathbf{p})$ denote the output of the computer $\mathcal{U}$ when presented with the program $\mathbf{p}$. Suppose $l(\mathbf{x})$ is the length of the string $\mathbf{x}$.

Then the Kolmogorov complexity $K_{\mathcal{U}}(\mathbf{x})$ of a string $\mathbf{x}$ with respect to a universal computer $\mathcal{U}$ is defined as

$$K_{\mathcal{U}}(\mathbf{x}) = \min_{\mathbf{p}:\, \mathcal{U}(\mathbf{p})=\mathbf{x}} l(\mathbf{p})$$

In other words: The *Kolmogorov complexity* of a string $\mathbf{x}$ with respect to computer $\mathcal{U}$ is the length of the shortest computer program that, when given to the computer $\mathcal{U}$ as input, results in the output $\mathbf{x}$.

The probability that a program $\mathbf{p}$ occurs randomly, i.e. by means of flipping a fair coin sequentially, is defined as $2^{-l(\mathbf{p})}$. (Note that thus shorter programs are considered much more probable than longer ones.) Then the *universal probability* of a string $\mathbf{x}$ is the probability that some program randomly drawn as a sequence of fair coin flips will print out the string $\mathbf{x}$,

$$P_{\mathcal{U}}(\mathbf{x}) = \sum_{\mathbf{p}:\, \mathcal{U}(\mathbf{p})=\mathbf{x}} 2^{-l(\mathbf{p})}$$

References

Cover, T.M.; and Thomas, J.A. (2006). Elements of Information Theory. (2. ed.) *Wiley, New York*. 1. ed. 1991.