The probability that a random ball is contained in a given ball

Appeared 2004 in: Journal of Interdisciplinary Mathematics **7(3)**, 337-340.

Uwe Saint-Mont

Some explanations and a small correction (2018)

The idea of this article is rather classical and what mathematicians call "straightforward:" What is the probability that an object, chosen at random, has a certain property? For example, Lewis Carroll chose a triangle in the plane "at random" and asked about the probability that it is obtuse.

I started with a circle $\mathbf{B}$ in the plane and asked the question in the title: If you first choose a point $C \in \mathbf{B}$ at random, and secondly point $A \in \mathbf{B}$, one thus defines a circle $B_r(C)$ around C with radius $r = AC$. What is the probability that $B_r(C)$ is a subset of $\mathbf{B}$?

Here is a related example in three-dimensional space: Planet earth is, at least approximately, a ball. Suppose the location C of an earthquake is random, and so is its magnitude (represented by some radius $r = AC$). What is the probability that this event will not reach the surface (i.e., the ball around C with radius r is entirely contained in the planet)?

The answer for n-dimensional space, and almost any metric (not just the Euclidean) is given in the article. Theorem 1 states that the probability we are looking for is

$$P(n) = \frac{(n!)^2}{(2n)!}$$

where $n! = n \cdot (n-1) \cdot (n-2) \cdots 1$. In particular, $P(1) = 1/2, P(2) = 2^2/(4 \cdot 3 \cdot 2 \cdot 1) = 1/6, P(3) = 6^2/(6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 1/20$, etc.

Qualitatively, this result was well-known: most of the mass of an n-dimensional ball is located near its surface. Thus, on the one hand, the probability that C is close to the surface is large. On the other hand, the probability that the distance $r = AC$ is quite large is also considerable, since A will typically not be close to C. Therefore, the larger n, the smaller the probability that the ball around C with radius AC will be contained in the n-dimensional ball, and $P(n)$ should be decreasing fast.

The second theorem extends this nice result to n-dimensional cuboids. The formulas in the proof are correct. However, the statement of the theorem should read:

Theorem 2. Let $0 < R_1 \leq \ldots \leq R_n$, and $T = T(R_1, \ldots, R_n)$ be an n-dimensional cuboid about the origin whose sides are $2R_i$ long. Successively, choose points C, A at random (according to the uniform) in this cuboid. If C determines the center of the n-dimensional random cube $B_n(C, AC) = \{x | d_\infty(C, x) \leq AC\}$, the probability that the latter cube is entirely contained in T is

$$\begin{aligned} P(n,T) \;=\; & \left(\prod_{m=1}^{n} v_m^2\right)\Bigg[n\beta(n,n+1) + (n-1)\left(\sum_{i=1}^{n} w_i\right)\beta(n-1,n+1) \\ & +(n-2)\left(\sum_{j<k} w_j w_k\right)\beta(n-2,n+1) + \ldots + \left(\prod_{l=2}^{n} w_l\right)\beta(1,n+1)\Bigg] \end{aligned}$$

where, for $y > 0$, $\beta(y, n+1)$ is the Beta-function, $\beta(y, n+1) = 0$ if $y \leq 0$, $v_i = R_1/R_i$, and $w_i = (1 - v_i)/v_i$.

AMS 2000 *subject classification.* Primary: 60D05, Secondary: 52A22.

Key words and Phrases: Random ball, Sampling at random, Geometric probability.