Statistical Tests

primitive → complicated → easy

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Basic Setup

Two tiers

General: $X \sim P_H$ (hypothetical law, hypothesis)

Specific: x (observation)

Fisher (1935): Every experiment may be said to exist only to give the facts a chance of disproving the null hypothesis.

Evaluating Observations

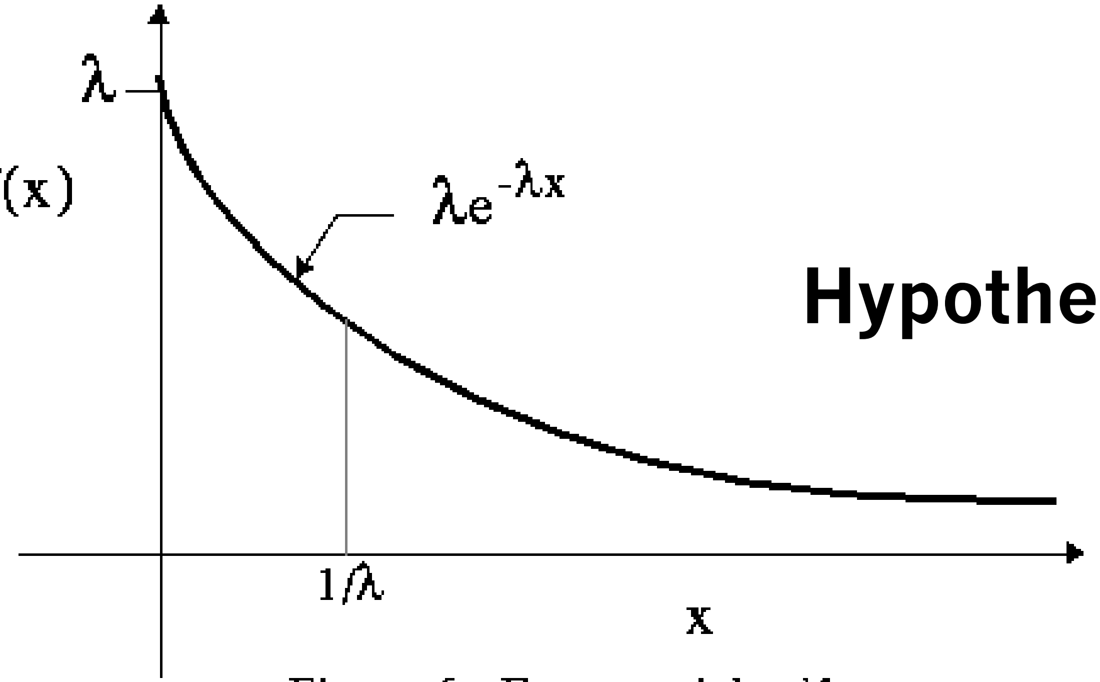


Figure 6. Exponential *pdf*

Hypothesis H

Evaluating Observations

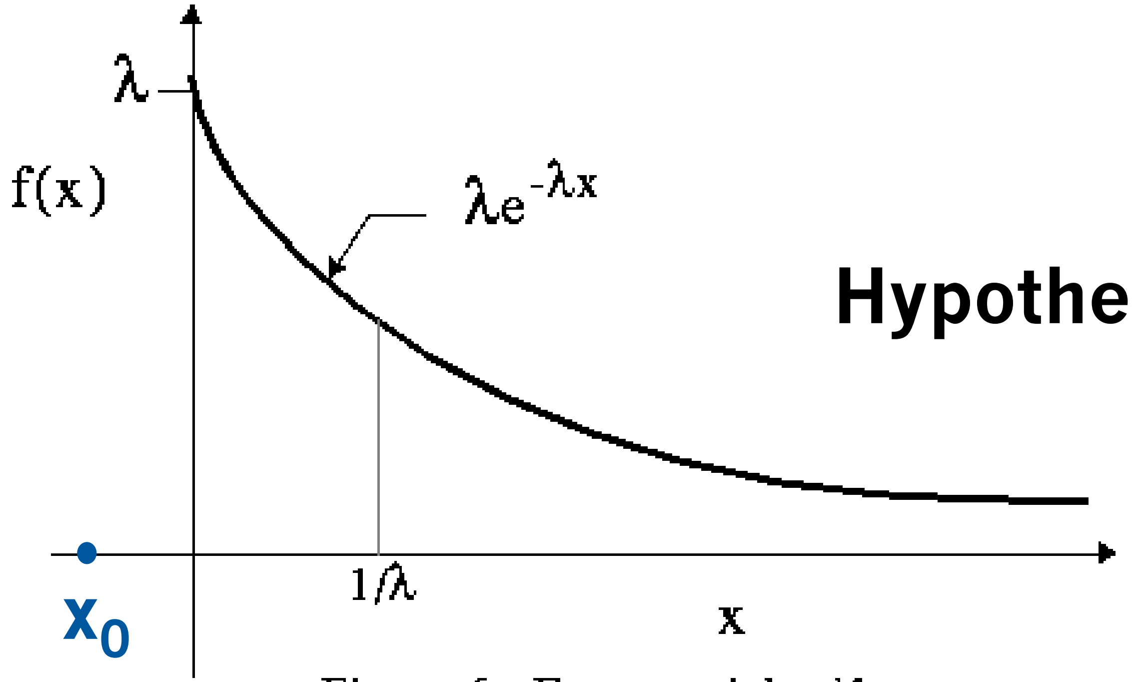


Figure 6. Exponential *pdf*

Hypothesis H

Impossible observation x_0

⇨ hypothesis must be wrong

Evaluating Observations

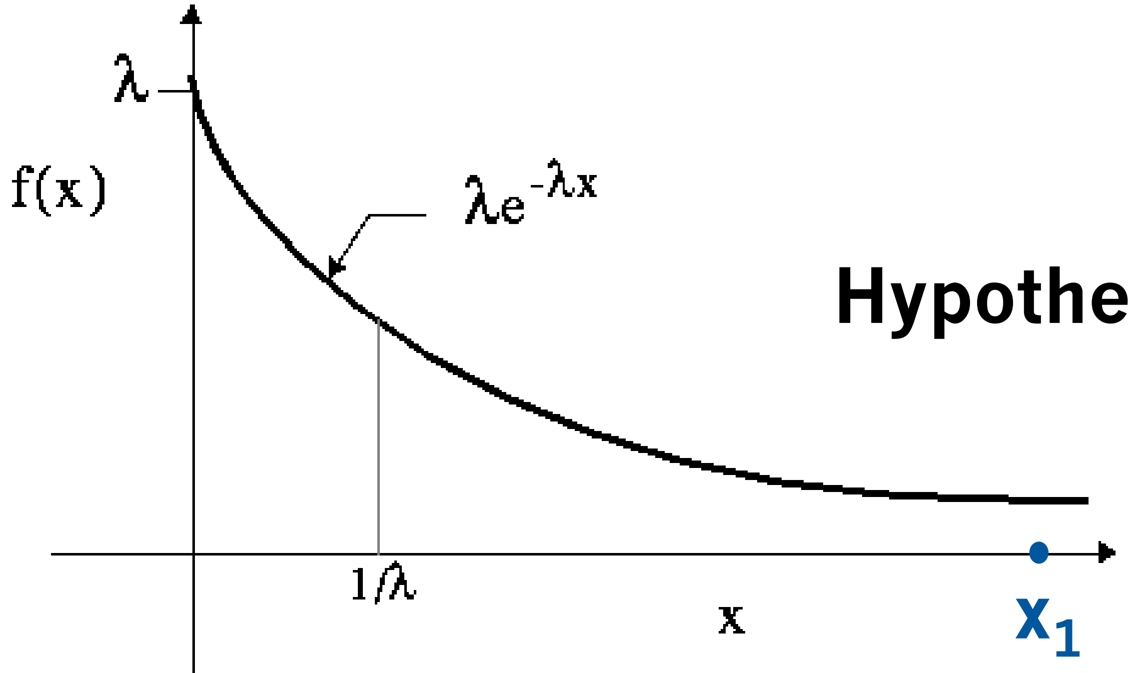


Figure 6. Exponential *pdf*

Hypothesis H

Possible observation x_1 ⇨ hypothesis may be wrong ⇨ rejection of H only "plausible"

Evaluating Observations

Two kinds of implausible observations:

$P_H(x_1)$ small, i.e., "an exceptionally rare chance has occurred" (Fisher 1956)

x_1 is an outlier, e.g., rather large for the hypothesis under consideration

Fisher‘s posterior test

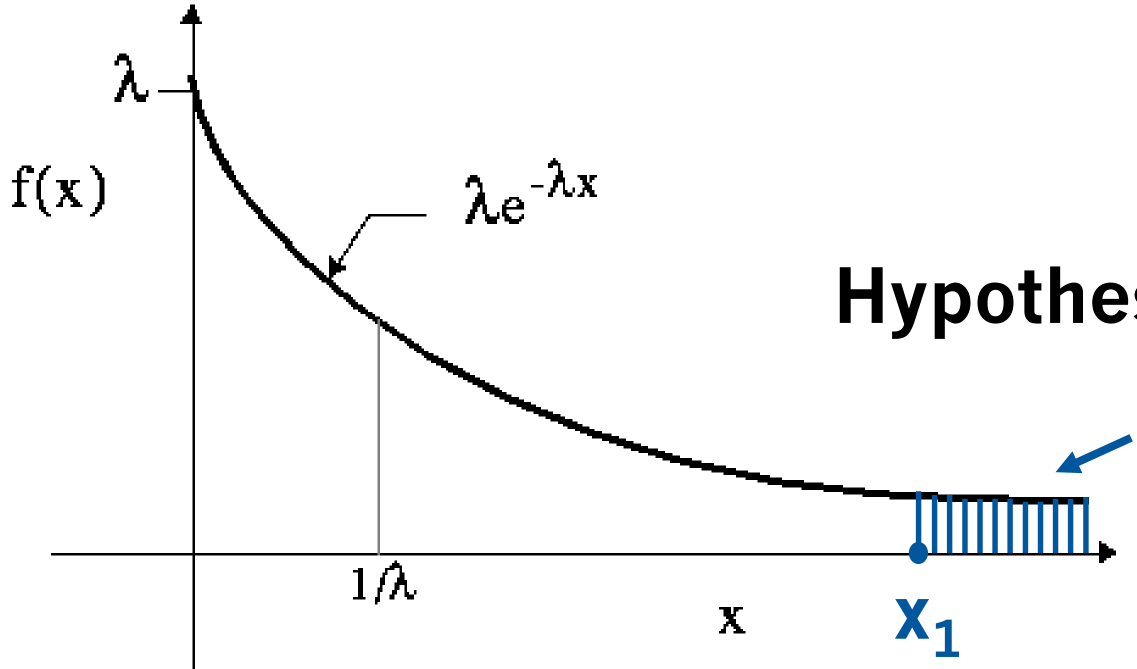


Figure 6. Exponential *pdf*

Hypothesis H

Given observation x_1 ⇨ compute $p = P_H(X \geq x_1)$

$$p = \int_{x_1}^{\infty} f(x)dx$$

Crucial Objection

Mathematically, the observed value x_1 is treated like all unobserved values $x > x_1$

Thus: An hypothesis that may be true is rejected because it has failed to predict observable results that have not occurred. This seems a remarkable procedure. (Jeffreys 1939)

Neyman&Pearson‘s prior test

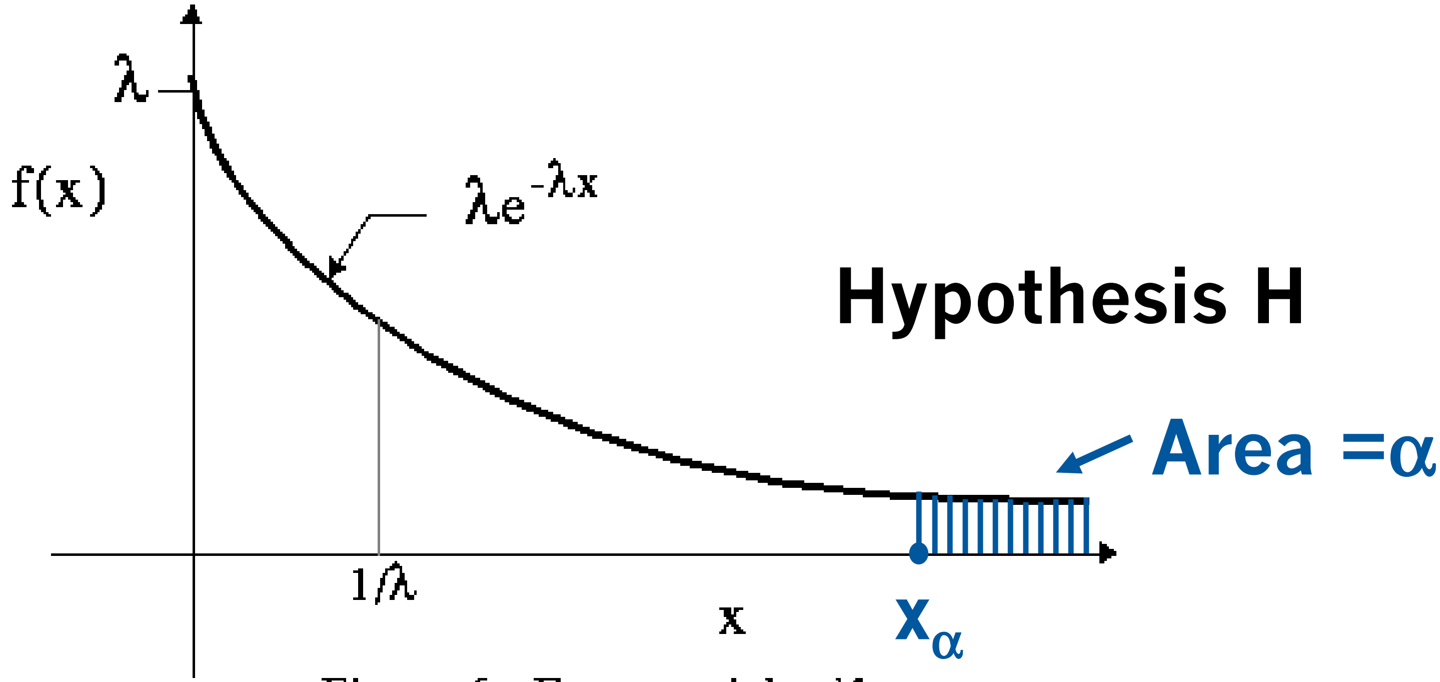


Figure 6. Exponential *pdf*

Given H, fix α such that $\alpha = \int_{x_\alpha}^{\infty} f(x)\,dx$

Reject H if $x > x_\alpha$ and do not reject H if $x \leq x_\alpha$

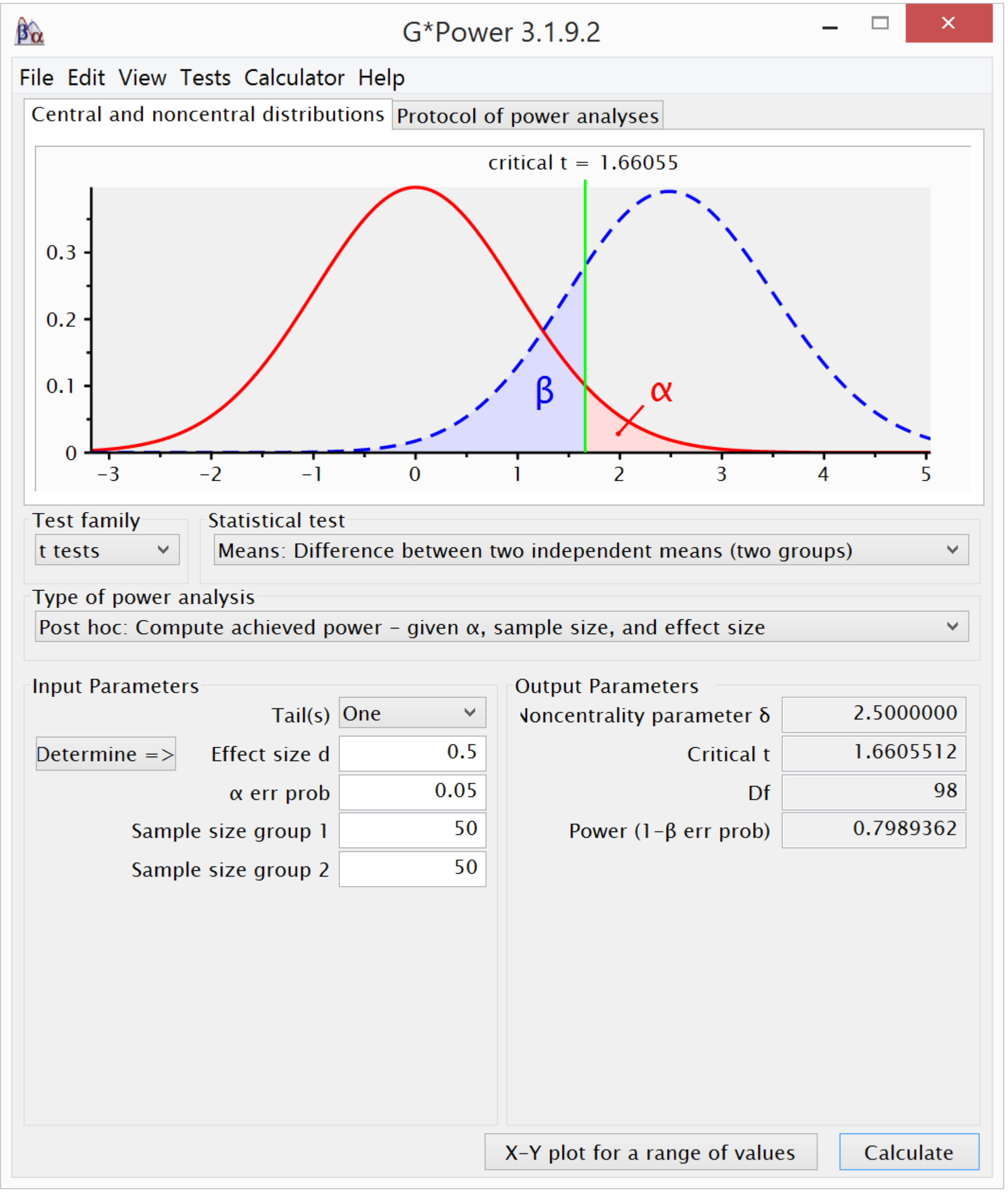
Objections

- Strict prior view ⇨ Dichotomous decision

- Information in the data is neglected

- Although the approaches are very different conceptionally, they are almost identical mathematically ⇨ Confusion

N&P's standard refined test



Given two hypotheses and the prior view, one has to deal with probabilities of error (α, β), the effect size, and sample sizes

Objections

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- **Invites "cherry picking", some kind of "ritual" (Gigerenzer), and further confusion: "alphabet soup" (Hubbard)**

- **Many further "side effects", in particular the idea that the smaller the sample the better. (Saint-Mont 2018)**

Main Objection

- Why use integrals as “proxies”?
- Given a hypothesis H and an alternative K, there is a
- Straightforward comparison:

$P_H(x)$ vs. $P_K(x)$

Likelihood: Results

- **Likelihood-ratio test (optimal):**

  decide in favour of H if $P_H(x) > P_K(x)$

  decide in favour of K if $P_K(x) > P_H(x)$

Likelihood: Results

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- **Bayes test: prior probs. of the hypos. Q_H, Q_K**

  decide in favour of H if $Q_H \, P_H(x) > Q_K \, P_K(x)$

  decide in favour of K if $Q_K \, P_K(x) > Q_H \, P_H(x)$

Testing: Look at the Distance

- **Straightforward link to information theory:**
  KL-divergence $D(K||H) = \int_{-\infty}^{\infty} P_K(x) log \frac{P_K(x)}{P_H(x)} dx$

- **Hoeffding's universal test (1965)**
  $P(X^n)$ empirical distribution of iid sample
  decide in favour of H if $D(P(X^n)||H) < c_n$

Elegant General Theory

- Likelihood function if there are many hyps.: decide in favour of the hypothesis H with the highest probability (maximum likelihood)

- Close connection with general criteria: Maximum Entropy, Minimum Message Length, Minimum Description Length, AIC, BIC,...

References

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Hoeffding, W. (1965). Asymptotically optimal tests for multinomial distributions. Ann. Math. Stat. 36, 369-408.

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Pearson, E. S. (1955). Statistical Concepts and their Relation to Reality. J. Royal Stat. Society, Ser. B 17(2), 204-207.

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Small Samples seem to be optimal

- **E.S. Pearson (1955):**

The appropriate test is one which, while involving through the choice of its significance level α only a very small risk of discarding my working hypothesis H prematurely will enable me to demonstrate with assurance $1-\beta$ (**but without any unnecessary amount of experimentation**) the reality of the influences which I suspect may be present.